Randomness and Hausdorff dimension

SIGMA Seminar, Fall 2023

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December 8th, 2023

University of Connecticut

Outline

Preliminaries

Definitions (from computability)

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A function $f : \mathbb{N} \to \mathbb{N}$ is a (partial) **computable** function if there exists an algorithm which computes the value of f on a given input.

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For this talk, you can think of computable functions and Turing machines as computer programs. Eventually, we want to ask the following question: *how much code is needed to describe certain programs?*

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We can then define a partial computable function on two arguments $U(e, x)$ such that $U(e, x) = \Phi_e(x)$. A function like this is said to be **universal**.

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Remark

If $f^A(n)\downarrow = a$, then we only use a finite initial segment σ of the oracle A to converge on n. That is, if $f^A(n)\downarrow = a$, then there exists a finite string $\sigma \subseteq A$ such that $f^\sigma(n)\downarrow = a$.

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In the literature, prefix-free Turing machines are sometimes referred to as **self-delimiting** machines.

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For a string (finite or infinite) σ, the **prefix-free Kolmogorov complexity** of σ is

$$K(\sigma) = \min\{|\tau| : \mathcal{U}(\tau) = \sigma\}.$$

Intuitively, the Kolmogorov complexity of σ tells you the length of the *shortest* program which describes the string σ.

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There are two other approaches to this question: *the measure-theoretic approach* (via null sets) and *the unpredictability approach* (via martingales).

A fantastic resource for algorithmic randomness is Downey and Hirschfeldt's book **Algorithmic Randomness and Complexity**. [DH10].

Notions of randomness

Chaitin's Ω

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$$K(A \upharpoonright n) \geq n - O(1).$$

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Here, $K^X(\sigma)$ is the Kolmogorov complexity based on a prefix-free universal Turing machine with oracle X, i.e., $\mathcal{U}^X$.

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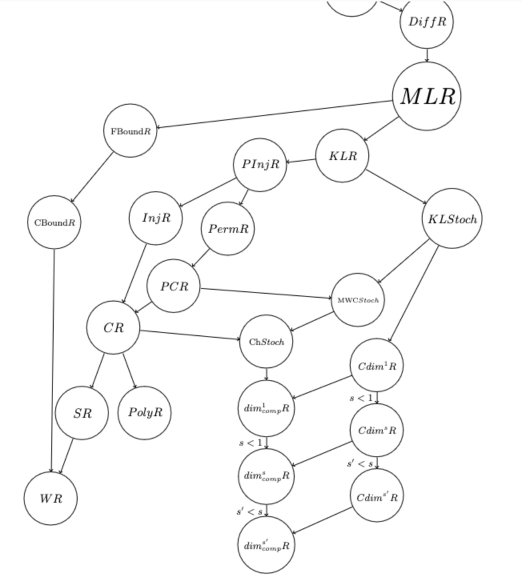
Definition ([MSU98])

A set is **Kolmogorov-Loveland random** if no partial computable nonmonotonic betting strategy succeeds on it.

Randomness zoo



Classical Hausdorff dimension

We begin with the following definitions, in the style of [LL18].

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Let $E \subseteq \mathbb{R}^n$. For $\delta > 0$, define $\mathcal{U}_\delta(E)$ to be the collection of all countable covers of E by sets of positive diameter at most δ. For $s \geq 0$, let

$$H^s_\delta(E) = \inf\left\{ \sum_{i\in\mathbb{N}} |U_i|^s : \{U_i\}_{i\in\mathbb{N}} \in \mathcal{U}_\delta(E) \right\}.$$

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The **s-dimensional Hausdorff outer measure** of E is

$$H^s(E) = \lim_{\delta\to 0^+} H^s_\delta(E).$$

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For this talk, we will use the characterization of effective Hausdorff dimension using initial segment complexity.

Effective Hausdorff dimension

Theorem ([May02])

For $A \in 2^\omega$*, the* **effective Hausdorff dimension** *of* A *is*

$$\dim(A) = \liminf_{n\to\infty} \frac{K(A \upharpoonright n)}{n}.$$

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This will be our definition of effective Hausdorff dimension.

An application to Falconer problems

From points to sets

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$$K_r(x) = \min\{K(q) : q \in \mathbb{Q}^n \cap B_{2^{-r}}(x)\}.$$

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Definition

For $x \in \mathbb{R}^n$, the **dimension** of x is

$$\dim(x) = \liminf_{r \to \infty} \frac{K_r(x)}{r}.$$

The point-to-set principle

We can relativize the definitions from the last slide to an arbitrary oracle $A \subseteq \mathbb{N}$ to define $K_r^A(x)$ and $\dim^A(x)$.

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Theorem (Point-to-set principle for Hausdorff dimension, [LL18])

For every set $E \subseteq \mathbb{R}^n$,

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x).$$

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The slogan: the existence of a high-dimensional point in a set $E \subseteq \mathbb{R}^n$ implies that E must have high dimension [LL18].

Conjecture (Falconer's conjecture, [Fal85],[Ios19])

Let $d \geq 2$. If the Hausdorff dimension of $E \subset \mathbb{R}^d$ is greater than $\frac{d}{2}$, then the Lebesgue measure of its distance set, $\Delta(E)$, is positive.

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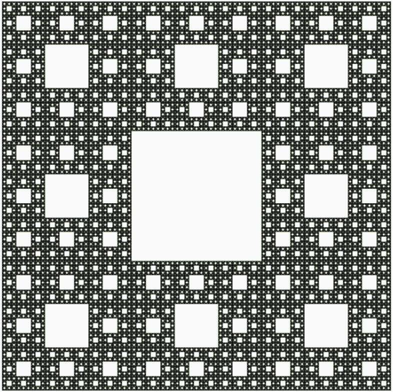
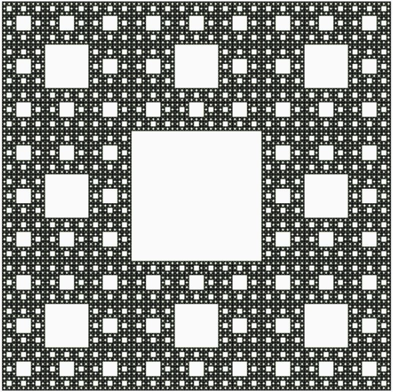
Intuitively: If a set $E \subset \mathbb{R}^d$ is a big set, then there should be lots of different ways to draw lines between any two points in E. So, they should have a wide range of different lengths.



The Sierpiński carpet after 6 steps



The Sierpiński carpet after 6 steps

The Hausdorff dimension of the carpet is $\dfrac{\log 8}{\log 3} \approx 1.8928.$

Results using the point-to-set principle

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Theorem ([Stu22])

Let $E \subseteq \mathbb{R}^2$ be an analytic set with Hausdorff dimension strictly greater than one. Then, for all $x \in \mathbb{R}^2$ outside a set of Hausdorff dimension at most 1,

$$\dim_H(\Delta_x E) \geq \frac{s}{4} + \frac{1}{2},$$

where $s = \dim_H(E)$.

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where $s = \dim_H(E)$.

$\Delta_x E$ is the **pinned distance set** of E, i.e.,
$\Delta_x E = \{|x - y| : y \in E\}$ where $E \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$.

Results using the point-to-set principle

The [Stu22] result is an improvement on the best known bounds for the pinned distance set problem for sets E whose dimension is close to 1.

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Another recent result about pinned distance sets using the point-to-set principle is in [FS23] by Fiedler and Stull for analytic sets in $\mathbb{R}^2$.

Most papers about Falconer's conjecture state that for $E \subseteq \mathbb{R}^n$, there exists a pin $x \in E$ where $\Delta_x E$ has large Hausdorff measure. The result in [FS23] shows that there are *a lot* of pins with this property in an analytic set E.

Thank You

Thanks for attending my talk! I'd be happy to answer any questions.

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