Computable categoricity relative to a c.e. degree

2024 North American Annual Meeting for the ASL

Java Darleen Villano

May 16th, 2024

University of Connecticut

Outline

Preliminaries

Definitions

Question

Given a structure $\mathcal{A}$ and a copy $\mathcal{B}$ of it, what is the complexity of the isomorphism between $\mathcal{A}$ and $\mathcal{B}$?

Definitions

Question

Given a structure $\mathcal{A}$ and a copy $\mathcal{B}$ of it, what is the complexity of the isomorphism between $\mathcal{A}$ and $\mathcal{B}$?

We restrict ourselves to countable structures with domain ω in a computable language.

Definitions

Question

Given a structure $\mathcal{A}$ and a copy $\mathcal{B}$ of it, what is the complexity of the isomorphism between $\mathcal{A}$ and $\mathcal{B}$?

We restrict ourselves to countable structures with domain ω in a computable language.

Definition

A computable structure $\mathcal{A}$ is **computably categorical** if for every computable copy $\mathcal{B}$ of $\mathcal{A}$, there exists a computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Examples of computable categoricity

Here are some examples of computably categorical structures.

Examples of computable categoricity

Here are some examples of computably categorical structures.

Example

- Computable linear orderings with only finitely many adjacent pairs (Remmel [Rem81]);

Examples of computable categoricity

Here are some examples of computably categorical structures.

Example

- Computable linear orderings with only finitely many adjacent pairs (Remmel [Rem81]);
- Computable algebraically closed fields of finite transcendence degree (Eršov [Erš77]); and

Examples of computable categoricity

Here are some examples of computably categorical structures.

Example

- Computable linear orderings with only finitely many adjacent pairs (Remmel [Rem81]);
- Computable algebraically closed fields of finite transcendence degree (Eršov [Erš77]); and
- Computable ordered groups of finite rank (Gončarov, Lempp, Solomon [GLS03]).

Examples of computable categoricity

Here are some examples of computably categorical structures.

Example

- Computable linear orderings with only finitely many adjacent pairs (Remmel [Rem81]);
- Computable algebraically closed fields of finite transcendence degree (Eršov [Erš77]); and
- Computable ordered groups of finite rank (Gončarov, Lempp, Solomon [GLS03]).

The given conditions in each example are both necessary and sufficient for computable categoricity.

Relativizing categoricity

The most studied relativization of categoricity is the following.

Relatively computably categorical

The most studied relativization of categoricity is the following.

Definition

A computable structure $\mathcal{A}$ is **relatively computably categorical** if for every copy (not necessarily computable) $\mathcal{B}$ of $\mathcal{A}$, there is a $\mathcal{B}$-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Relatively computably categorical

The most studied relativization of categoricity is the following.

Definition

A computable structure $\mathcal{A}$ is **relatively computably categorical** if for every copy (not necessarily computable) $\mathcal{B}$ of $\mathcal{A}$, there is a $\mathcal{B}$-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Remark

If a computable structure is relatively computably categorical, then it is computably categorical.

Relatively computably categorical

The most studied relativization of categoricity is the following.

Definition

A computable structure $\mathcal{A}$ is **relatively computably categorical** if for every copy (not necessarily computable) $\mathcal{B}$ of $\mathcal{A}$, there is a $\mathcal{B}$-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Remark

If a computable structure is relatively computably categorical, then it is computably categorical.

The converse is not true in general.

Algebraic characterization of computable categoricity

For a class of structures, if there is a purely algebraic characterization of computable categoricity, then a computably categorical structure $\mathcal{A}$ will often also be relatively computably categorical.

Algebraic characterization of computable categoricity

For a class of structures, if there is a purely algebraic characterization of computable categoricity, then a computably categorical structure $\mathcal{A}$ will often also be relatively computably categorical.

The connection between an algebraic characterization of computable categoricity and the equivalence of plain and relativized computable categoricity was clarified by the following result.

Algebraic characterization of computable categoricity

For a class of structures, if there is a purely algebraic characterization of computable categoricity, then a computably categorical structure $\mathcal{A}$ will often also be relatively computably categorical.

The connection between an algebraic characterization of computable categoricity and the equivalence of plain and relativized computable categoricity was clarified by the following result.

Theorem (Ash, Knight, Manasse, and Slaman [Ash+89]; Chisholm [Chi90])

A structure is relatively computably categorical if and only if it has a formally Σ_1 Scott family.

Plain but not relative

Gončarov [Gon77] built the first example of a structure which was computably categorical but *not* relatively computably categorical, using an enumeration result due to Selivanov [Sel76].

Plain but not relative

Gončarov [Gon77] built the first example of a structure which was computably categorical but *not* relatively computably categorical, using an enumeration result due to Selivanov [Sel76].

He also later showed if that if a computable structure had a single 2-decidable copy, then it must be relatively computably categorical.

Plain but not relative

Gončarov [Gon77] built the first example of a structure which was computably categorical but *not* relatively computably categorical, using an enumeration result due to Selivanov [Sel76].

He also later showed if that if a computable structure had a single 2-decidable copy, then it must be relatively computably categorical.

Theorem (Gončarov [Gon80])

If a structure is computably categorical and its $\forall\exists$ theory is decidable, then it is relatively computably categorical.

Plain but not relative

Gončarov [Gon77] built the first example of a structure which was computably categorical but *not* relatively computably categorical, using an enumeration result due to Selivanov [Sel76].

He also later showed if that if a computable structure had a single 2-decidable copy, then it must be relatively computably categorical.

Theorem (Gončarov [Gon80])

If a structure is computably categorical and its $\forall\exists$ theory is decidable, then it is relatively computably categorical.

Kudinov [Kud96] showed that the assumption of 2-decidability could not be lowered to 1-decidability.

Using a finite number of jumps

We now move beyond computable isomorphisms by allowing ourselves a fixed number of jumps.

Using a finite number of jumps

We now move beyond computable isomorphisms by allowing ourselves a fixed number of jumps.

Definition

For α a computable ordinal, a computable structure $\mathcal{A}$ is Δ^0_α**-categorical** if for any computable copy $\mathcal{B}$ of $\mathcal{A}$, there is a Δ^0_α-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Using a finite number of jumps

We now move beyond computable isomorphisms by allowing ourselves a fixed number of jumps.

Definition

For α a computable ordinal, a computable structure $\mathcal{A}$ is Δ^0_α-**categorical** if for any computable copy $\mathcal{B}$ of $\mathcal{A}$, there is a Δ^0_α-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Definition

A computable structure $\mathcal{A}$ is **relatively** Δ^0_α-**categorical** if for any copy $\mathcal{B}$ of $\mathcal{A}$, there is a $\Delta^0_\alpha(\mathcal{B})$-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Plain and relative Δ^0_α-categoricity

There are some examples of classes of structures where plain and relative Δ^0_α-categoricity coincide.

Plain and relative Δ^0_α-categoricity

There are some examples of classes of structures where plain and relative Δ^0_α-categoricity coincide.

Theorem (Bazhenov [Baz14])

A computable Boolean algebra $\mathcal{B}$ is Δ^0_2-categorical if and only if it is relatively Δ^0_2-categorical.

Plain and relative Δ^0_α-categoricity

There are some examples of classes of structures where plain and relative Δ^0_α-categoricity coincide.

Theorem (Bazhenov [Baz14])

A computable Boolean algebra $\mathcal{B}$ is Δ^0_2-categorical if and only if it is relatively Δ^0_2-categorical.

Similar to the case for computable categoricity and relative computable categoricity, plain and relative Δ^0_α-categoricity need not coincide. There are several examples in a paper by Fokina, Harizanov, and Turetsky [FHT19] (trees of finite and infinite heights, etc.).

Categoricity relative to a degree

A newer relativization

The following relativization of categoricity appears in the main result of a paper by Downey, Harrison-Trainor, and Melnikov [DHTM21].

A newer relativization

The following relativization of categoricity appears in the main result of a paper by Downey, Harrison-Trainor, and Melnikov [DHTM21].

Definition

For $X \in 2^{\mathbb{N}}$, a computable structure $\mathcal{A}$ is **computably categorical relative to a degree X** if for every X-computable copy $\mathcal{B}$ of $\mathcal{A}$, there is an X-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

A newer relativization

The following relativization of categoricity appears in the main result of a paper by Downey, Harrison-Trainor, and Melnikov [DHTM21].

Definition

For $X \in 2^{\mathbb{N}}$, a computable structure $\mathcal{A}$ is **computably categorical relative to a degree X** if for every X-computable copy $\mathcal{B}$ of $\mathcal{A}$, there is an X-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Fact

A computable structure $\mathcal{A}$ is **relatively computably categorical** *if for all $X \in 2^{\mathbb{N}}$, $\mathcal{A}$ is computably categorical relative to X.*

The cone above $0''$

The following is known.

The cone above $\mathbf{0}''$

The following is known.

Fact (Downey, Harrison-Trainor, Melnikov [DHTM21])

If $\mathcal{A}$ is a computable structure and it is computably categorical relative to some degree $\mathbf{d} \geq \mathbf{0}''$, then $\mathcal{A}$ has a $\mathbf{0}''$-computable Σ_1^0 Scott family. So, $\mathcal{A}$ is computably categorical relative to all $\mathbf{d} \geq \mathbf{0}''$.

The cone above $\mathbf{0}''$

The following is known.

Fact (Downey, Harrison-Trainor, Melnikov [DHTM21])

If $\mathcal{A}$ is a computable structure and it is computably categorical relative to some degree $\mathbf{d} \geq \mathbf{0}''$, then $\mathcal{A}$ has a $\mathbf{0}''$-computable Σ^0_1 Scott family. So, $\mathcal{A}$ is computably categorical relative to all $\mathbf{d} \geq \mathbf{0}''$.

So at $\mathbf{0}''$ and above, any computable structure $\mathcal{A}$ will settle on whether it is computably categorical relative to all degrees or to none of them.

The cone above $\mathbf{0}''$

The following is known.

Fact (Downey, Harrison-Trainor, Melnikov [DHTM21])

If $\mathcal{A}$ is a computable structure and it is computably categorical relative to some degree $\mathbf{d} \geq \mathbf{0}''$, then $\mathcal{A}$ has a $\mathbf{0}''$-computable Σ^0_1 Scott family. So, $\mathcal{A}$ is computably categorical relative to all $\mathbf{d} \geq \mathbf{0}''$.

So at $\mathbf{0}''$ and above, any computable structure $\mathcal{A}$ will settle on whether it is computably categorical relative to all degrees or to none of them.

Question

What happens between $\mathbf{0}$ and $\mathbf{0}''$?

In the c.e. degrees

In the c.e. degrees, being computably categorical relative to a degree is *not* monotonic.

In the c.e. degrees

In the c.e. degrees, being computably categorical relative to a degree is *not* monotonic.

Theorem (Downey, Harrison-Trainor, Melnikov [DHTM21])

There is a computable structure $\mathcal{A}$ and c.e. degrees $\mathbf{0} = Y_0 <_T X_0 <_T Y_1 <_T X_1 <_T \dots$ such that

(1) *$\mathcal{A}$ is computably categorical relative to Y_i for each i,*

(2) *$\mathcal{A}$ is not computably categorical relative to X_i for each i,*

(3) *$\mathcal{A}$ is computably categorical relative to $\mathbf{0}'$.*

Partial orders of c.e. degrees

We extend this result to partial orders of c.e. degrees.

Partial orders of c.e. degrees

We extend this result to partial orders of c.e. degrees.

Theorem (V. [Vil24])

Let $P = (P, \leq)$ be a computable partially ordered set and let $P = P_0 \sqcup P_1$ be a computable partition. Then, there exists a computable directed graph $\mathcal{G}$ and an embedding h of P into the c.e. degrees where

(1) *$\mathcal{G}$ is computably categorical;*

(2) *$\mathcal{G}$ is computably categorical relative to each degree in $h(P_0)$; and*

(3) *$\mathcal{G}$ is not computably categorical relative to each degree in $h(P_1)$.*

Basic ideas in the proof

We have a priority construction with four types of requirements based on four goals:

Basic ideas in the proof

We have a priority construction with four types of requirements based on four goals:

(1) embedding P into the c.e. degrees;

Basic ideas in the proof

We have a priority construction with four types of requirements based on four goals:

(1) embedding P into the c.e. degrees;

(2) making the graph $\mathcal{G}$ computably categorical;

Basic ideas in the proof

We have a priority construction with four types of requirements based on four goals:

(1) embedding P into the c.e. degrees;
(2) making the graph $\mathcal{G}$ computably categorical;
(3) making $\mathcal{G}$ computably categorical relative to all degrees in $h(P_0)$;

Basic ideas in the proof

We have a priority construction with four types of requirements based on four goals:

(1) embedding P into the c.e. degrees;

(2) making the graph $\mathcal{G}$ computably categorical;

(3) making $\mathcal{G}$ computably categorical relative to all degrees in $h(P_0)$; and

(4) making $\mathcal{G}$ not computably categorical relative to any degree in $h(P_1)$.

Basic ideas in the proof

We have a priority construction with four types of requirements based on four goals:

(1) embedding P into the c.e. degrees;
(2) making the graph $\mathcal{G}$ computably categorical;
(3) making $\mathcal{G}$ computably categorical relative to all degrees in $h(P_0)$; and
(4) making $\mathcal{G}$ not computably categorical relative to any degree in $h(P_1)$.

We use a tree of strategies to organize restraints and parameters.

Future directions: in the generic degrees

Definition

A degree **d** is **low for isomorphism** if for every pair of computable structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \cong_{\mathbf{d}} \mathcal{B}$ if and only if $\mathcal{A} \cong_{\Delta_1^0} \mathcal{B}$.

Future directions: in the generic degrees

Definition

A degree $\mathbf{d}$ is **low for isomorphism** if for every pair of computable structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \cong_{\mathbf{d}} \mathcal{B}$ if and only if $\mathcal{A} \cong_{\Delta^0_1} \mathcal{B}$.

Theorem (Franklin, Solomon [FS14])

Every 2-generic degree is low for isomorphism.

Future directions: in the generic degrees

Definition

A degree $\mathbf{d}$ is **low for isomorphism** if for every pair of computable structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \cong_{\mathbf{d}} \mathcal{B}$ if and only if $\mathcal{A} \cong_{\Delta^0_1} \mathcal{B}$.

Theorem (Franklin, Solomon [FS14])

Every 2-generic degree is low for isomorphism.

This means that there *cannot* be a computable structure $\mathcal{A}$ which is not computably categorical but is computably categorical relative to $\mathbf{d}$ for a 2-generic degree $\mathbf{d}$.

Future directions: in the generic degrees

Definition

A degree **d** is **low for isomorphism** if for every pair of computable structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \cong_{\mathbf{d}} \mathcal{B}$ if and only if $\mathcal{A} \cong_{\Delta^0_1} \mathcal{B}$.

Theorem (Franklin, Solomon [FS14])

Every 2-generic degree is low for isomorphism.

This means that there *cannot* be a computable structure $\mathcal{A}$ which is not computably categorical but is computably categorical relative to **d** for a 2-generic degree **d**.

Conjecture

There exists a 1-generic G such that there is a computable directed graph $\mathcal{A}$ where $\mathcal{A}$ is not computably categorical but is computably categorical relative to G.

Future directions: identifying pathological behavior in classes of structures

Question

For structures other than directed graphs, can you produce an example which witnesses the pathological behavior in the poset result?

Future directions: identifying pathological behavior in classes of structures

Question

For structures other than directed graphs, can you produce an example which witnesses the pathological behavior in the poset result?

There are some results in the literature that give a negative result for certain classes of structures already.

Future directions: identifying pathological behavior in classes of structures

Question

For structures other than directed graphs, can you produce an example which witnesses the pathological behavior in the poset result?

There are some results in the literature that give a negative result for certain classes of structures already.

Theorem (Bazhenov [Baz14])

For every degree $\mathbf{d} < \mathbf{0}'$, a computable Boolean algebra is $\mathbf{d}$-computably categorical if and only if it is computably categorical.

Future directions: identifying pathological behavior in classes of structures

Conjecture

For the following classes of structures, there exists a computable example in each class which witnesses the pathological behavior in the poset result: symmetric, irreflexive graphs; partial orderings; lattices; rings with zero-divisors; integral domains of arbitrary characteristic; commutative semigroups; and 2-step nilpotent groups.

Future directions: identifying pathological behavior in classes of structures

Conjecture

For the following classes of structures, there exists a computable example in each class which witnesses the pathological behavior in the poset result: symmetric, irreflexive graphs; partial orderings; lattices; rings with zero-divisors; integral domains of arbitrary characteristic; commutative semigroups; and 2-step nilpotent groups.

This is based on the codings given in a paper by Hirschfeldt, Khoussainov, Shore, and Slinko in [Hir+02]. In this paper, they specified codings which satisfied certain conditions and thus preserved several computability theoretic properties of structures, such as the degree spectrum or computable dimension.

References

[Ash+89] C. Ash et al. **"Generic copies of countable structures".** *Annals of Pure and Applied Logic* 42.3 (1989), pp. 195–205. ISSN: 0168-0072.

[Baz14] N. A. Bazhenov. **"Δ^0_2-Categoricity of Boolean Algebras".** *Journal of Mathematical Sciences* 203.4 (2014), pp. 444–454.

[Chi90] J. Chisholm. **"Effective Model Theory vs. Recursive Model Theory".** *The Journal of Symbolic Logic* 55.3 (1990), pp. 1168–1191. ISSN: 00224812.

[DHTM21] R. Downey, M. Harrison-Trainor, and A. Melnikov. **“Relativizing computable categoricity”.** *Proc. Amer. Math. Soc.* 149.9 (2021), pp. 3999–4013. ISSN: 0002-9939.

[Erš77] J. L. Erš. **“Theorie Der Numerierungen III”.** *Mathematical Logic Quarterly* 23.19-24 (1977), pp. 289–371.

[FHT19] E. Fokina, V. Harizanov, and D. Turetsky. **“Computability-theoretic categoricity and Scott families”.** *Annals of Pure and Applied Logic* 170.6 (2019), pp. 699–717. ISSN: 0168-0072.

[FS14] J. N. Y. Franklin and R. Solomon. **“Degrees that Are Low for Isomorphism”.** *Computability* 3 (2014), pp. 73–89.

[GLS03] S. S. Gončarov, S. Lempp, and R. Solomon. **"The computable dimension of ordered abelian groups".** *Advances in Mathematics* 175.1 (2003), pp. 102–143.

[Gon77] S. S. Gončarov. **"The quantity of nonautoequivalent constructivizations".** *Algebra and Logic* 16.3 (May 1977), pp. 169–185. ISSN: 1573-8302.

[Gon80] S. S. Gončarov. **"The problem of the number of nonautoequivalent constructivizations".** *Algebra i Logika* 19.6 (1980), pp. 621–639, 745.

[Hir+02] D. R. Hirschfeldt et al. **"Degree spectra and computable dimensions in algebraic structures".** *Annals of Pure and Applied Logic* 115.1 (2002), pp. 71–113. ISSN: 0168-0072. DOI: https://doi.org/10.1016/S0168-0072(01)00087-2. URL: https://www.sciencedirect.com/science/article/pii/S0168007201000872.

[Kud96] O. V. Kudinov. **"An autostable 1-decidable model without a computable Scott family of $\exists$-formulas".** *Algebra and Logic* 35.4 (July 1996), pp. 255–260. ISSN: 1573-8302.

[Rem81] J. B. Remmel. **"Recursively Categorical Linear Orderings".** *Proc. Amer. Math. Soc.* 83.2 (1981), pp. 387–391.

[Sel76] V. L. Selivanov. **"Enumerations of families of general recursive functions".** *Algebra and Logic* 15.2 (Mar. 1976), pp. 128–141. ISSN: 1573-8302.

[Vil24] J. D. Villano. **Computable categoricity relative to a c.e. degree.** 2024. arXiv: 2401.06641 [math.LO].

Thanks

Thank you for your attention!

I'd be happy to answer any questions.